

VSR symmetries in the DKP algebra: the interplay between Dirac and Elko spinor fields

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VSR symmetries are here naturally incorporated in the DKP algebra on the spin-0 and the spin-1 DKP sectors. We show that the Elko (dark) spinor fields structure plays an essential role on accomplishing this aim, unravelling hidden symmetries on the bosonic DKP fields under the action of discrete symmetries.

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I. INTRODUCTION

Elko spinor fields (dual-helicity eigenspinors of the charge conjugation operator [1, 2]) are spin-1/2 matter fields, with unexpected properties that make them prime candidates to describe dark matter. Recent efforts to scrutinize the underlying structure of Elko fields, that incorporate both the Very Special Relativity (VSR) paradigm [3] and dark matter as well, have been reported in [4]. In this context, an Elko spinor mass generation mechanism was proposed in [5], via a natural coupling to the kink solution of a $\lambda\phi^4$ field theory, regarding exotic couplings between Elko spinor fields and scalar field topological solutions [5]. Some attempts to detect Elko at the LHC have been moreover proposed [6], as well as promising applications [7–11].

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Elko spinor fields occupy just one type among other classes of spinor fields that encompass the well known Dirac, Weyl, Majorana, flag-poles, dipoles, and flag-dipoles ones [12, 13]. Elko has provided prominent applications in cosmology, gravity [8, 9, 22–29]), field theory [4, 5, 18–20] and its further supersymmetric formulation as well [21]. Those classes have been shown worth to further explore, and the phenomenology related to new fermionic fields has been recently investigated [6, 30]. Moreover, in the framework of $f(R)$ and ESK gravity, the Dirac equation admits new solutions distinct from Dirac spinor fields, which are flag-dipoles spinor fields [31, 32]. Black hole thermodynamics has been also investigated in the context of Elko spinor fields [33].

The aim of this article is to evince hidden symmetries that underlie bosonic fields described by the spacetime DKP (Duffin-Kemmer-Petiau) algebra. This algebra is here a subalgebra of a bigger space formed by the tensor product of two Clifford algebras that comprise both Dirac and Elko spinor fields. The unexpected behavior of DKP spinor fields under discrete symmetries show that the full Lorentz group is not the symmetry group associated to the whole DKP algebra. Instead, merely the subgroups $\text{HOM}(2)$ and $\text{SIM}(2)$ are related to VSR symmetries¹. Rephrasing it, although Cohen and Glashow state that VSR implies special relativity either in the context of local quantum field theory or in the framework of CP conservation [3], merging VSR with Elko is still feasible [14]. The amplitude for the two body decay of a spinless particle at rest may depend on the direction of the decay products relative to the VSR preferred direction [3, 14], and VSR signature is expected to arise for the mass dimension one Fermi field [6]. While a new connection with the VSR has begun to be evinced [4, 14], we stress that the $\text{HOM}(2)$ is both sufficient and necessary to encompass the negative outcome of the Michelson-Morley experiment [3].

This paper is organized as follows: in the next Section we review the basic features of the DKP algebra and present it as the product of two Clifford algebras, in such a way that DKP fields can be presented as the tensor product between spinor fields. In Section III, Elko spinor fields are briefly revisited. The types of spinor fields (under Lounesto spinor field classification) that are preserved under sum of spinors, are discussed in Section IV. Moreover, the conditions that the Elko and Dirac spinor fields must satisfy for the correct definition of the DKP fields, in terms of the tensor product of spinor fields, are obtained. This is accomplished in order to assure that the spinor fields classes, under the Lounesto spinor field classification, are preserved. Elko and Dirac spinors are shown to be building blocks of the DKP algebra, by revealing how the Elko unusual features reflects at

¹ The incorporation of the parity and the time reversal operators widens the VSR subgroups to the full Lorentz group.

the bosonic level of the symmetries of the DKP algebra. The DKP algebra can be obtained by the tensor product between Dirac spinor fields, and thus preserves Lorentz symmetries. The VSR symmetries, underlying the Elko spinor fields structure [3, 4], are thus induced on the DKP algebra. In the last Section we conclude.

II. THE DKP ALGEBRA

The study of elementary particles by means of classical wave theory depends on the charged (either electric charge or moment dipolar charge) or uncharged aspect of the particle under consideration. A particle of mass m is said to be a meson if it is described by the wave equation in Minkowski spacetime

$$(\eta^{\mu\nu} \beta_\mu \partial_\nu + im) \psi = 0, \quad (1)$$

where β_μ satisfy the commutation rules proposed in [34–36]:

$$\beta_\mu \beta_\nu \beta_\rho + \beta_\rho \beta_\nu \beta_\mu = \eta_{\nu\rho} \beta_\mu + \eta_{\nu\mu} \beta_\rho. \quad (2)$$

In addition, the Dirac equation is well known to be written as

$$(\eta^{\mu\nu} \gamma_\mu \partial_\nu + im) \psi = 0, \quad (3)$$

denoting the set $\{\gamma_\mu\}$ of Dirac matrices in Eq.(15), and the set $\{\mathbf{1}, \gamma_\mu, \gamma_\mu \gamma_\nu, \gamma_\mu \gamma_\nu \gamma_\rho, \gamma_0 \gamma_1 \gamma_2 \gamma_3\}$ ($\mu, \nu, \rho = 0, 1, 2, 3$, and $\mu < \nu < \rho$) as a basis for $M(4, \mathbb{C})$, such that $\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\eta_{\mu\nu} \mathbf{1}$. The Clifford product is denoted by juxtaposition.

The Dirac equation is usually written in a form that can be immediately compared with the meson wave equation, by the prescription $\beta_\mu \mapsto \gamma_\mu$. The massless DKP theory can not be obtained as a zero mass limit of the massive DKP case. Some authors consider the Harish-Chandra Lagrangian density for the massless DKP theory in Minkowski spacetime, given by [41]

$$\mathcal{L} = i\bar{\psi} \tau \beta^\mu \partial_\mu \psi - i\partial_\mu \bar{\psi} \beta^\mu \tau \psi - \bar{\psi} \tau \psi, \quad (4)$$

where τ is a singular idempotent matrix satisfying

$$\beta^\mu \tau + \tau \beta^\mu = \beta^\mu, \quad \text{and} \quad \tau^\dagger = \tau, \quad (5)$$

and moreover $\beta^{0\dagger} = \beta^0$, $\beta^{i\dagger} = -\beta^i$. Besides, the angular momentum is provided by $S^{\mu\nu} = \beta^\mu \beta^\nu - \beta^\nu \beta^\mu$ [34–36].

By denoting hereon a $n \times n$ matrix B by \mathbb{B}_n , the spin-0 sector of the DKP algebra is realized when a specific representation of the DKP algebra is used, $\tau = \{0\} \oplus \mathbb{I}_4$. It thus reproduces the massless Klein-Gordon-Fock field [42]. The scalar sector of the massless DKP theory associated to the 5-dimensional representation of massless DKP algebra provides

$$\psi = (\varphi, A^\mu)^\top, \quad (6)$$

where φ and A^μ are scalar and 4-vector under Lorentz transformations, respectively.

The spin-1 DKP field ψ can be provided by choosing $\tau = \mathbb{O}_4 \oplus \mathbb{I}_6$, therefore the DKP field is thus a 10-component column vector

$$\psi = (\psi^\mu, F^{\mu\nu})^\top, \quad (7)$$

where ψ^μ and $F^{\mu\nu}$ are respectively a 4-vector and some antisymmetric tensor in Minkowski space-time. In other words, the above Lagrangian yields to the DKP wave equation

$$\eta^{\mu\nu} \beta_\mu \partial_\nu \psi - \tau \psi = 0. \quad (8)$$

In order to settle on the spin 0 sector the so called Umezawa's projectors P and P^μ [43] are applied on Eq.(8), and taking into account the above relations (5) implying $\tau P = P \tau$ and $P^\mu \tau + \tau P^\mu = P^\mu$, the equation of motion for the massless scalar field $P\psi$ reads $\partial_\mu \partial^\mu (P\psi) = 0$. Usually, the gauge invariance is derived when a representation for β^μ is employed in which

$$\tau = \text{diag}(\alpha, 1 - \alpha, 1 - \alpha, 1 - \alpha, 1 - \alpha). \quad (9)$$

In this representation the one-column DKP wave function and its projections are given by

$$\psi = (\varphi, \psi^\mu)^\top, \quad P\psi = (\varphi, \mathbb{O})^\top, \quad P\tau\psi = (\alpha\varphi, \mathbb{O})^\top, \quad (10)$$

$$P^\mu\psi = (\psi^\mu, \mathbb{O})^\top, \quad P^\mu\tau\psi = ((1 - \alpha)\psi^\mu, \mathbb{O})^\top \quad (11)$$

where $\mathbb{O} \equiv [0]_{4 \times 1}$. The condition $\tau^2 - \tau = 0$ implies for the α parameter that $\alpha^2 - \alpha = 0 \Rightarrow \alpha = 0, 1$. The value $\alpha = 1$ corresponds to a topological field, while $\alpha = 0$ reproduces the massless Klein-Gordon field [41]. In this representation the explicit relations among the components of the massless spin 0 DKP field are $(1 - \alpha)\psi^\mu = i\partial^\mu\varphi$ and $\alpha\varphi = i\partial_\mu\psi^\mu$. For $\alpha = 0$ the DKP Lagrangian density (4) reduces to the usual one for the massless Klein-Gordon field $\mathcal{L} = \partial^\mu\varphi^*\partial_\mu\varphi$. For more aspects on the equivalence between DKP and the Klein-Gordon see, for instance, Refs. [44, 45]. Moreover, an interesting approach of the Dirac and DKP equations for spin 1/2 and spins 0 and 1, respectively, as the simplest special cases of the Bhabha system of first order relativistic wave

equations for arbitrary spin can be found at [37, 38], and references therein. The DKP algebra has been approached furthermore in other interesting contexts [39, 40].

The DKP field can be related to the tensor product of two Dirac spinor fields, and endows their symmetries under charge conjugation \mathcal{C} , parity \mathcal{P} , and time reversal \mathcal{T} . Although in [46] the A^μ and the $F^{\mu\nu}$ are respectively interpreted as the electromagnetic potential and the field strength, the formalism is thoroughly general. It means that for the standard DKP field, the charge conjugation, parity, and time reversal are involutions, when acting on the scalar, the 4-vector, and the antisymmetric tensor of the DKP field components in (6, 7). In particular, $\mathcal{P}^2 = \mathbb{I} = (\mathcal{C}\mathcal{P}\mathcal{T})^2$.

DKP fields given by Eqs. (6) and (7) can be expressed as the tensor product of two algebraic spinor fields, namely, elements of a minimal left ideal in the Clifford-Dirac algebra $\mathbb{C}\ell_{1,3}$. In order to realize the relationship between Clifford algebras and DKP algebras [47, 48], let us take a quadratic space (V, g) , which can be thought as being the tangent space at a point on a Lorentzian manifold. Let us also consider \mathcal{A} an associative algebra with unity $1_{\mathcal{A}}$ and let γ be the linear application $\gamma : V \rightarrow \mathcal{A}$. The pair (\mathcal{A}, γ) is a Clifford algebra $\mathcal{C}\ell(V, g)$ for the quadratic space (V, g) when \mathcal{A} is generated as an algebra by $\{\gamma(v) \mid v \in V\}$ and $\{a1_{\mathcal{A}} \mid a \in \mathbb{R}\}$, satisfying $\gamma(v)\gamma(u) + \gamma(u)\gamma(v) = 2g(v, u)1_{\mathcal{A}}$, for all $v, u \in V$. For a basis $\{e_\mu\}$ of V , the element $\gamma(e_\mu)$ is usually denoted by γ_μ [12].

Throughout this paper V is considered to be the Minkowski spacetime $\mathbb{R}^{1,3}$. The Clifford-Dirac algebra for this particular case is denoted by $\mathbb{C}\ell_{1,3}$. Spinors are well known to be elements of a minimal ideal of $\mathbb{C}\ell_{1,3}$ [12]. The composed spinor describing an element of the DKP algebra is therefore an element of $\mathbb{C}\ell_{1,3} \otimes \mathbb{C}\ell_{1,3}$. By considering the mapping $\delta : \mathbb{R}^{1,3} \rightarrow \mathbb{C}\ell_{1,3} \otimes \mathbb{C}\ell_{1,3}$ defined by

$$\delta(v) = \frac{1}{2}(v \otimes 1 + 1 \otimes v) \quad (12)$$

(here "1" denotes the identity in $\mathbb{C}\ell_{1,3}$), the property

$$\delta(u)\delta(v)\delta(u) = \frac{1}{8}(uvu \otimes 1 + u^2 \otimes v + (uv + vu) \otimes u + u \otimes (uv + vu) + v \otimes u^2 + 1 \otimes uvu) \quad (13)$$

can be derived [48]. The Clifford relation $uv + vu = 2g(u, v)1$ implies $uvu = 2g(u, v)u - g(u, u)v$ and consequently

$$\delta(u)\delta(v)\delta(u) = g(u, v)\delta(u).$$

By the universal property of the spacetime DKP algebra $B_{1,3}$, the application δ extends to an algebra monomorphism $\Delta : B_{1,3} \rightarrow \mathbb{C}\ell_{1,3} \otimes \mathbb{C}\ell_{1,3}$ [47], mapping every $\psi \in B_{1,3}$ in $\frac{1}{2}(\psi \otimes 1 + 1 \otimes \psi)$.

It is worth mentioning that, for every $v \in \mathbb{R}^{1,3}$, $\Delta(2v^2 - g(v, v)) = v \otimes v$. The DKP algebra $B_{1,3}$ is thus the subalgebra of $\mathbb{C}\ell_{1,3} \otimes \mathbb{C}\ell_{1,3}$ generated by all elements $\delta(v)$ [47, 48].

DKP fields are usually written as the tensor product of two Dirac spinor fields. Hereupon we shall depart from this assumption, and evince the unexpected role of introducing Elko spinor fields as the DKP fields building blocks. We prove that the interplay between Dirac and Elko spinor fields manifest VSR symmetries in the spin-0 and spin-1 DKP algebra sectors, with the aid of the graded tensor product. Firstly, let us brief revisit Elko spinor fields.

III. ELKO SPINOR FIELDS

In this Section some properties of Elko spinor fields are briefly revisited. An Elko can be expressed in general as [1, 14]

$$\lambda(k^\mu) = \begin{pmatrix} i\Theta\phi^*(k^\mu) \\ \phi(k^\mu) \end{pmatrix}, \quad k^\mu \equiv \lim_{p \rightarrow 0} (m, \mathbf{p}), \quad (14)$$

where $\phi(k^\mu)$ denotes a left-handed Weyl spinor and $p = \|\mathbf{p}\|$. Given the rotation generators denoted by \mathcal{J} , the Wigner's spin-1/2 time reversal operator Θ satisfies $\Theta \mathcal{J} \Theta^{-1} = -\mathcal{J}^*$. Hereon, as in [1], the Weyl representation of γ^μ

$$\gamma^0 = \begin{pmatrix} \mathbb{O}_2 & \mathbb{I}_2 \\ \mathbb{I}_2 & \mathbb{O}_2 \end{pmatrix}, \quad \gamma^k = \begin{pmatrix} \mathbb{O}_2 & -\sigma_k \\ \sigma_k & \mathbb{O}_2 \end{pmatrix} \quad (15)$$

is used, where σ_i are the Pauli matrices. Elko spinor fields are eigenspinors of the charge conjugation operator C , namely, $C\lambda(k^\mu) = \pm\lambda(k^\mu)$. The plus [minus] sign regards self-conjugate [anti self-conjugate] spinor fields, denoted by $\lambda^S(k^\mu)$ [$\lambda^A(k^\mu)$]. Explicitly, the complete form of Elko spinor fields can be found by solving the equation of helicity $(\sigma \cdot \hat{\mathbf{p}})\phi^\pm(k^\mu) = \pm\phi^\pm(k^\mu)$ in the rest frame and subsequently performing a boost [1, 14]. Therefore Elko spinor fields are given by [14]

$$\lambda_\pm^S(p^\mu) = \sqrt{\frac{E+m}{2m}} \left(1 \mp \frac{p}{E+m} \right) \lambda_\pm^S(k^\mu), \quad \lambda_\pm^A(p^\mu) = \sqrt{\frac{E+m}{2m}} \left(1 \pm \frac{p}{E+m} \right) \lambda_\pm^A(k^\mu), \quad (16)$$

where $\lambda_\pm^S(k^\mu) = \begin{pmatrix} i\Theta[\phi^\pm(k^\mu)]^* \\ \phi^\pm(k^\mu) \end{pmatrix}$ and $\lambda_\pm^A(k^\mu) = \pm \begin{pmatrix} -i\Theta[\phi^\mp(k^\mu)]^* \\ \phi^\mp(k^\mu) \end{pmatrix}$. Moreover, the notation

$$\phi_L^+(k^\mu) = \sqrt{m} \begin{pmatrix} \cos(\theta/2) \exp(-i\phi/2) \\ \sin(\theta/2) \exp(+i\phi/2) \end{pmatrix}, \quad \phi_L^-(k^\mu) = \sqrt{m} \begin{pmatrix} -\sin(\theta/2) \exp(-i\phi/2) \\ \cos(\theta/2) \exp(+i\phi/2) \end{pmatrix}$$

is used [14].

There are several interesting and unusual aspects concerning Elko theory. Most importantly for our purposes here is that the underlying Elko properties are ruled by VSR subgroups. Cohen and Glashow argued that Very Special Relativity, rather than Special Relativity, could be the fundamental symmetry of nature at the planck scale, with the standard model emerging as an effective theory [3]. Ahluwalia and Horvath, however, proposed that VSR could appear at the standard model scale, related to dark matter [4].

It is well known that when the parity operator is absent in a given relativistic theory, it is possible to rebuilt the dynamical objects by thinking of irreducible representations of subgroups of the Lorentz group [3]. The situation is the following: by removing the parity operator from the full Lorentz group (therefore working out of the discrete symmetries scope) it is possible to rearrange some of the boosts and rotations generators, giving rise to subgroups other than the orthochronous proper one. There are four possible groups, but we are mainly interested in the so called homotheties and similitude groups, denoted by $\text{HOM}(2)$ and $\text{SIM}(2)$ respectively.

Explicitly, given \mathcal{J} and \mathcal{K} the rotation and boost generators, the generators $T_1 := K_x + J_y$ and $T_2 := K_y - J_x$ can be defined. Adding K_z to those generators yields the algebra $\mathfrak{hom}(2)$, which is a 3-parameter algebra associated with the group of homotheties, $\text{HOM}(2)$. On the other hand, adding J_z to $\mathfrak{hom}(2)$ lead to a wider algebra associated to the similitude group, $\text{SIM}(2)$. VSR is realized taking $\text{HOM}(2)$ and $\text{SIM}(2)$ as the symmetry groups of the theory. In fact, Cohen and Glashow have shown that the group $\text{HOM}(2)$, which is a subgroup of $\text{SIM}(2)$, is necessary and sufficient to [3]: a) explain the results of the Michelson-Morley experiments and its more sensitive results; b) ensure that the speed of light is the same for all observers; c) preserve SR time dilatation and the law of velocity addition. Furthermore the Elko theory (more precisely, the Elko spin sums) are invariant under the action of $\text{HOM}(2)$ and covariant under $\text{SIM}(2)$ [4].

An enlightening review of VSR and why this symmetry must emerge in Elko theory can be found in the reference [4]. For VSR in a wider class of Lorentz violation theory see [15].

In what follows, we shall investigate the typical signature of VSR subgroups in DKP algebra by using Elko spinor fields as building blocks of the DKP field. Notice, however, that in this case we must focus our attention naturally in the action of the charge conjugation operator on the spinor fields.

IV. ELKO AND DIRAC SPINOR FIELDS: THE DKP ALGEBRA

This Section is devoted to introduce class-preserving Elko and Dirac spinor fields – under Lounesto spinor field classification – in order to express DKP fields in terms of the tensor product between Elko and Dirac spinor fields. It implies that Elko spinor fields can manifest different properties on the spin-0 and spin-1 sectors of DKP fields. Besides, the DKP algebra is shown to have an unexpected behavior, when expressed as the graded tensor product between Elko and Dirac spinor fields.

Let us start by considering the Minkowski spacetime M and $\{\mathbf{e}_\mu\}$ a section of the frame bundle $\mathbf{P}_{\text{SO}_{1,3}^\epsilon}(M)$ and $\{\theta^\mu\}$ be respectively its associated dual basis. Classical spinor fields carrying a $D^{(1/2,0)} \oplus D^{(0,1/2)}$ representation of $\text{SL}(2, \mathbb{C}) \simeq \text{Spin}_{1,3}^\epsilon$ are sections of the vector bundle $\mathbf{P}_{\text{Spin}_{1,3}^\epsilon}(M) \times_\rho \mathbb{C}^4$, where ρ stands for the $D^{(1/2,0)} \oplus D^{(0,1/2)}$ representation of $\text{SL}(2, \mathbb{C})$ in the complex 4×4 matrices. Within this formalism, it is possible to make use of the multivector structure and write down a mother spinor field given by

$$\psi \sim (\sigma + \mathbf{J} + i\mathbf{S} - i\gamma_{0123}\mathbf{K} + \gamma_{0123}\omega)\eta, \quad (17)$$

where η is a spinor and $\sigma, \mathbf{J}, \mathbf{S}, \mathbf{K}$, and ω are the bilinear covariants provided by:

$$\begin{aligned} \sigma &= \psi^\dagger \gamma_0 \psi, \quad \mathbf{J} = J_\mu \theta^\mu = \psi^\dagger \gamma_0 \gamma_\mu \psi \theta^\mu, \quad \mathbf{S} = S_{\mu\nu} \theta^{\mu\nu} = \frac{1}{2} \psi^\dagger \gamma_0 i \gamma_{\mu\nu} \psi \theta^\mu \wedge \theta^\nu, \\ \mathbf{K} &= K_\mu \theta^\mu = \psi^\dagger \gamma_0 i \gamma_{0123} \gamma_\mu \psi \theta^\mu, \quad \omega = -\psi^\dagger \gamma_0 \gamma_{0123} \psi. \end{aligned} \quad (18)$$

Not every combination of bilinear covariants generate different types of spinors, since the bilinear covariants satisfy the Fierz identities:

$$\mathbf{K}^2 + \mathbf{J}^2 = 0 = \mathbf{J} \cdot \mathbf{K}, \quad \mathbf{J}^2 = \omega^2 + \sigma^2, \quad \mathbf{K} \wedge \mathbf{J} = (\omega + \sigma \gamma_{0123})\mathbf{S}. \quad (19)$$

It turns out that the Lounesto spinor field classification provides just the following spinor field classes [12], where in the first three classes clearly $\mathbf{J}, \mathbf{S}, \mathbf{K} \neq 0$:

- | | |
|---|--|
| 1) $\sigma \neq 0, \quad \omega \neq 0$ | 4) $\sigma = 0 = \omega, \quad \mathbf{K} \neq 0, \quad \mathbf{S} \neq 0$ |
| 2) $\sigma \neq 0, \quad \omega = 0$ | 5) $\sigma = 0 = \omega, \quad \mathbf{K} = 0, \quad \mathbf{S} \neq 0$ |
| 3) $\sigma = 0, \quad \omega \neq 0$ | 6) $\sigma = 0 = \omega, \quad \mathbf{K} \neq 0, \quad \mathbf{S} = 0$ |

Lounesto spinor field classes are well known not be preserved by sum of spinor fields. For instance, the sum of two Weyl spinor fields (type-(6)) are usually Dirac spinor fields (that are encompassed

by type-(1), -(2), and -(3)) under this classification. We shall analyze the space generated by tensor product of Elko linear spaces and Dirac linear spaces. Thus, in what follows we aim to find the vector space structure underlying Elko and Dirac spinor fields spaces. This is necessary to guarantee that the linear combination of spinor fields in the same class under Lounesto classification is still in the same spinor field class. In order to accomplish this for Elko spinor fields, we need to find a linear structure such that every spinor field must be either a type-(5) spinor field or the zero vector. Therefore an eigenspinor of the charge conjugation operator restricts to those corresponding to real eigenvalues. A natural way to find this structure is considering the eigenspaces of the charge conjugation operator associated to positive or negative eigenvalues, namely, the self-conjugate class preserving and anti self-conjugate class preserving spaces, respectively. They are 4-dimensional real linear spaces:

$$\mathcal{E}^S = \text{span}_{\text{lin}}^{\mathbb{R}} \{(-i, 0, 0, 1)^\top, (-1, 0, 0, i)^\top, (0, i, 1, 0)^\top, (0, 1, i, 0)^\top\}, \quad (20)$$

$$\mathcal{E}^A = \text{span}_{\text{lin}}^{\mathbb{R}} \{(i, 0, 0, 1)^\top, (1, 0, 0, i)^\top, (0, -i, 1, 0)^\top, (0, -1, i, 0)^\top\}. \quad (21)$$

Explicit calculations can show that merely a few combinations mixing elements of both spaces are of type-(5), but they do not correspond to Elko.

For Dirac spinor fields (types-(1),-(2),-(3)), at least σ or ω must be non null. Given two Dirac spinors

$$\psi(x) = (\psi_1(x), \psi_2(x), \psi_3(x), \psi_4(x))^\top, \quad \zeta(x) = (\zeta_1(x), \zeta_2(x), \zeta_3(x), \zeta_4(x))^\top,$$

where $\psi_\mu(x)$ and $\zeta_\mu(x)$ are complex functions, each one of the following conditions ensure that the sum of Dirac [Elko] spinors are Dirac [Elko] spinors. In fact, for the sum of type-(1) Dirac spinors be type-(1) Dirac spinors, the conditions below must hold:

$$-\psi_1^* \psi_3 - \psi_2^* \psi_4 + \psi_3^* \psi_1 - \psi_4^* \psi_2 \neq -\zeta_1^* \zeta_3 + \zeta_2^* \zeta_4 + \zeta_3^* \zeta_1 - \zeta_4^* \zeta_2, \quad (22)$$

$$\pm \psi_1^* \psi_3 \pm \psi_2^* \psi_4 + \psi_3^* \psi_1 \pm \psi_4^* \psi_2 \neq \zeta_1^* \zeta_3 \pm \zeta_2^* \zeta_4 + \zeta_3^* \zeta_1 \pm \zeta_4^* \zeta_2, \quad (23)$$

$$\pm \psi_1^* \psi_3 \pm \psi_2^* \psi_4 \pm \psi_3^* \psi_1 \pm \psi_4^* \psi_2 \neq \zeta_1^* \zeta_3 \pm \zeta_2^* \zeta_4 + \zeta_3^* \zeta_1 \pm \zeta_4^* \zeta_2, \quad (24)$$

$$-\psi_1^* \psi_4 \pm \psi_2^* \psi_3 \mp \psi_3^* \psi_2 + \psi_4^* \psi_1 \neq -\zeta_1^* \zeta_4 \pm \zeta_2^* \zeta_3 \mp \zeta_3^* \zeta_2 + \zeta_4^* \zeta_1, \quad (25)$$

$$\psi_1^* \psi_4 \pm \psi_2^* \psi_3 + \psi_3^* \psi_2 \pm \psi_4^* \psi_1 \neq \zeta_1^* \zeta_4 \pm \zeta_2^* \zeta_3 + \zeta_3^* \zeta_2 \pm \zeta_4^* \zeta_1, \quad (26)$$

$$\|\psi_1\|^2 \pm \|\psi_2\|^2 \mp \|\psi_3\|^2 - \|\psi_4\|^2 \neq \|\zeta_1\|^2 \pm \|\zeta_2\|^2 \mp \|\zeta_3\|^2 - \|\zeta_4\|^2. \quad (27)$$

For type-(2) Dirac spinors, one of the conditions in (22, 23, 24), namely

$$\psi_1^* \psi_3 + \psi_2^* \psi_4 + \psi_3^* \psi_1 + \psi_4^* \psi_2 \neq \zeta_1^* \zeta_3 + \zeta_2^* \zeta_4 + \zeta_3^* \zeta_1 + \zeta_4^* \zeta_2,$$

can be relaxed, as well as one of the conditions in (27)

$$\|\psi_1\|^2 + \|\psi_2\|^2 - \|\psi_2\|^2 - \|\psi_4\|^2 \neq \|\zeta_1\|^2 + \|\zeta_2\|^2 - \|\zeta_2\|^2 - \|\zeta_4\|^2,$$

for type-(3) Dirac spinor fields. Actually, the conditions for the sum of the spinors $\psi(x)$ and $\zeta(x)$ to be a Dirac (regular) spinor can be summarized in the expression

$$(\psi_1 + \zeta_1)(\psi_3 + \zeta_3)^* \neq (\psi_2 + \zeta_2)(\psi_4 + \zeta_4)^*. \quad (28)$$

These relations are also satisfied by two 4-dimensional complementary linear spaces:

$$\mathcal{D}^+ = \text{span}_{\mathbb{R}}^{\mathbb{R}} \{(1, 0, 1, 0)^\top, (i, 0, i, 0)^\top, (0, i, 0, 1)^\top, (0, -1, 0, i)^\top\}, \quad (29)$$

$$\mathcal{D}^- = \text{span}_{\mathbb{R}}^{\mathbb{R}} \{(-1, 0, 1, 0)^\top, (-i, 0, i, 0)^\top, (0, -i, 0, 1)^\top, (0, 1, 0, i)^\top\}. \quad (30)$$

These spaces contain all types of Dirac spinor fields. For Dirac fields the constraints $\mathbf{K} \neq 0$ or $\mathbf{S} \neq 0$ are not necessary: Lounesto classification implies that there are no classes of spinors with σ or ω non null, and \mathbf{K} or \mathbf{S} null. In fact, it results from the implications $\mathbf{K} = 0 \Rightarrow \sigma = 0 = \omega$ and $\mathbf{S} = 0 \Rightarrow \sigma = 0 = \omega$. For an useful characterization of the spinor classes under Lounesto classification see [16], with a survey in [17]. Such classification in spaces endowed with arbitrary bilinear forms is accomplished in [26].

Hereon we shall denote by $\mathcal{D}[\mathcal{E}]$ any of the spaces (29, 30) [(20, 21)]. Operators acting on the space $\mathcal{S}_1 \otimes \mathcal{S}_2$, where $\mathcal{S}_i \in \{\mathcal{D}, \mathcal{E}\}$, $i = 1, 2$, are constructed as follows:

$$\mathcal{O} = \frac{1}{\sqrt{2}}(\mathcal{O}_{\mathcal{S}_1} \otimes \mathbb{I}_{\mathcal{S}_2} + \mathbb{I}_{\mathcal{S}_1} \otimes \mathcal{O}_{\mathcal{S}_2}), \quad (31)$$

where $\mathcal{O}_{\mathcal{S}_i} \in \text{End}(\mathcal{S}_i)$ denotes general operators in the endomorphism group acting on either the Dirac or Elko class preserving spinor fields space. We choose the above factor $\frac{1}{\sqrt{2}}$ in order that the discrete symmetries act on the tensor product of spinor fields as involutions, for some cases. Hereupon our notation is devoid of the subindexes $(\cdot)_{\mathcal{S}_i}$, $(\cdot)_{\mathcal{D}}$, or $(\cdot)_{\mathcal{E}}$, which shall be implicitly noticeable.

Given $\chi_i \in \mathcal{S}_i$, the successive action of the operator $\mathcal{O} \in \text{End}(\mathcal{S}_1 \otimes \mathcal{S}_2)$ respectively given by

$$\mathcal{O}(\chi_1 \otimes \chi_2) = \frac{1}{\sqrt{2}}(\mathcal{O}\chi_1 \otimes \chi_2 + \chi_1 \otimes \mathcal{O}\chi_2), \quad (32)$$

$$\mathcal{O}^2(\chi_1 \otimes \chi_2) = \frac{1}{2}\mathcal{O}^2\chi_1 \otimes \chi_2 + \mathcal{O}\chi_1 \otimes \mathcal{O}\chi_2 + \frac{1}{2}\chi_1 \otimes \mathcal{O}^2\chi_2. \quad (33)$$

In general it is straightforward to show that

$$\mathcal{O}^n(\chi_1 \otimes \chi_2) = 2^{-\frac{n}{2}} \sum_{p=0}^n \binom{n}{p} \mathcal{O}^{n-p} \chi_1 \otimes \mathcal{O}^p \chi_2.$$

The commutator and the anti-commutator of two operators $\mathcal{O}_i = \frac{1}{\sqrt{2}}(\mathcal{O}_i \otimes \mathbb{I} + \mathbb{I} \otimes \mathcal{O}_i)$, acting on the space $\mathcal{S}_1 \otimes \mathcal{S}_2$ are thus provided by

$$[\mathcal{O}_1, \mathcal{O}_2] = \frac{1}{2}([\mathcal{O}_1, \mathcal{O}_2] \otimes \mathbb{I} + \mathbb{I} \otimes [\mathcal{O}_1, \mathcal{O}_2]), \quad (34)$$

$$\{\mathcal{O}_1, \mathcal{O}_2\} = \frac{1}{2}(\{\mathcal{O}_1, \mathcal{O}_2\} \otimes \mathbb{I} + 2\mathcal{O}_1 \otimes \mathcal{O}_2 + 2\mathcal{O}_2 \otimes \mathcal{O}_1 + \mathbb{I} \otimes \{\mathcal{O}_1, \mathcal{O}_2\}). \quad (35)$$

As Dirac and Elko spinor fields are elements of minimal left ideals in the Clifford-Dirac algebra $\mathbb{C}\ell_{1,3}$, they are not a priori even elements, under the graded involution that defines the \mathbb{Z}_2 -grading of $\mathbb{C}\ell_{1,3}$. Therefore, the graded tensor product can be defined. Actually, in the four dimensional space-time the map given in the equation (12) does not define a morphism when we change the standard tensor product to the graded one. When the domain is restricted, we it does define a morphism, in particular in the spinor operator context.

The alternating tensor product $\mathbb{C}\ell_{1,3} \hat{\otimes} \mathbb{C}\ell_{1,3}$ is the algebra generated by the product $a \hat{\otimes} b$, defined by

$$(a_1 \hat{\otimes} b_1)(a_2 \hat{\otimes} b_2) = (-1)^{\deg(b_1) \deg(a_2)} a_1 a_2 \hat{\otimes} b_1 b_2, \quad a_1, a_2, b_1, b_2 \in \mathbb{C}\ell_{1,3}. \quad (36)$$

Hence the analogue of Eq.(32) in this context provided by the graded tensor product reads

$$\begin{aligned} \mathcal{O}(\chi_1 \hat{\otimes} \chi_2) &= \frac{1}{\sqrt{2}}[(\mathcal{O} \hat{\otimes} \mathbb{I})(\chi_1 \hat{\otimes} \chi_2) + (\mathbb{I} \hat{\otimes} \mathcal{O})(\chi_1 \hat{\otimes} \chi_2)] \\ &= \frac{1}{\sqrt{2}}[\mathcal{O} \chi_1 \hat{\otimes} \chi_2 + \chi_1 \hat{\otimes} \mathcal{O} \chi_2]. \end{aligned} \quad (37)$$

This case and the former one presented in (32) are similar. Notwithstanding, it does not hold in general, when higher order compositions of \mathcal{O} are regarded. Indeed

$$\begin{aligned} \mathcal{O}^2(\chi_1 \hat{\otimes} \chi_2) &= \frac{1}{\sqrt{2}} \mathcal{O}(\mathcal{O} \chi_1 \hat{\otimes} \chi_2 + \chi_1 \hat{\otimes} \mathcal{O} \chi_2) \\ &= \frac{1}{2}[\mathcal{O}^2 \chi_1 \hat{\otimes} \chi_2 + \mathcal{O} \chi_1 \hat{\otimes} \mathcal{O} \chi_2 + (-1)^{\deg \mathcal{O}} \mathcal{O} \chi_1 \hat{\otimes} \mathcal{O} \chi_2 + \chi_1 \hat{\otimes} \mathcal{O}^2 \chi_2]. \end{aligned} \quad (38)$$

The parity, charge conjugation, and time reversion operators are respectively defined as [1]

$$\mathcal{P} = e^{i\phi} \gamma^0 \mathcal{R}, \quad \mathcal{C} = i\gamma^2 \mathcal{K}, \quad \mathcal{T} = i\gamma^1 \gamma^3 \mathcal{C}, \quad (39)$$

where in spherical coordinates the action of the operator \mathcal{R} is $\{\theta \mapsto \pi - \theta, \phi \mapsto \phi + \pi, r \mapsto r\}$. The operator \mathcal{K} is the complex conjugation operator. At this point it is worth to call attention to the

fact that although spinors live in $\mathbb{C}\ell_{1,3}$, the operators are defined on the representation space and need to be written in terms of $\mathbb{C}\ell_{1,3}$. Here, solely \mathcal{K} need to be adapted, by acting on an algebraic spinor φ as $\mathcal{K}\varphi = \gamma^{013}\varphi^*(\gamma^{013})^{-1}$ [12]. Consequently, \mathcal{C}, \mathcal{P} , and \mathcal{T} are all odd operators under the graded involution, and therefore

$$2\mathcal{C}^2 = \mathcal{C}^2 \hat{\otimes} \mathbb{I} + \mathbb{I} \hat{\otimes} \mathcal{C}^2, \quad (40)$$

$$2\mathcal{P}^2 = \mathcal{P}^2 \hat{\otimes} \mathbb{I} + \mathbb{I} \hat{\otimes} \mathcal{P}^2, \quad (41)$$

$$2\mathcal{T}^2 = \mathcal{T}^2 \hat{\otimes} \mathbb{I} + \mathbb{I} \hat{\otimes} \mathcal{T}^2, \quad (42)$$

hold for the alternating tensor product.

In order to analyze the behavior of class preserving spaces above obtained under actions of combinations of $\mathcal{C}, \mathcal{P}, \mathcal{T}$ operators, we will explicitly compute, in the next sections, the action of these operators on all combinations for the tensor product between Dirac spinors. As Elko spinor fields manifest VSR symmetries, solely the action of the operator \mathcal{C} is taken into account in Subsections IV.B and IV.C all possible actions arising upon such a possibility.

A. The tensor product between Dirac spinor fields

Starting with the parity operator $\mathcal{P} \in \text{End}(\mathcal{D} \otimes \mathcal{D})$, Eq.(33) implies

$$\begin{aligned} \mathcal{P}^2(\psi_1 \otimes \psi_2) &= \frac{1}{2}(\mathcal{P}^2\psi_1 \otimes \psi_2 + 2\mathcal{P}\psi_1 \otimes \mathcal{P}\psi_2 + \psi_1 \otimes \mathcal{P}^2\psi_2) \\ &= \psi_1 \otimes \psi_2 + \mathcal{P}\psi_1 \otimes \mathcal{P}\psi_2, \end{aligned} \quad (43)$$

since the parity operator acting twice on a Dirac spinor is the identity operator, i. e., $\mathcal{P}_{\mathcal{D}}^2 = \mathbb{I}_{\mathcal{D}}$. Therefore, for all $n = 1, 2, \dots$, there is a periodicity mod 2 for the operator $\mathcal{P} \in \text{End}(\mathcal{D} \otimes \mathcal{D})$, given by

$$\mathcal{P}^{2n-1} = 2^{n-3/2}(\mathcal{P} \otimes \mathbb{I} + \mathbb{I} \otimes \mathcal{P}), \quad \mathcal{P}^{2n} = 2^{n-1}(\mathbb{I} \otimes \mathbb{I} + \mathcal{P} \otimes \mathcal{P}). \quad (44)$$

Moreover, the anti-commutators and commutators of the charge conjugation and the parity operators in the DKP algebra are respectively given by

$$\begin{aligned} \{\mathcal{C}, \mathcal{P}\} &= \mathcal{C} \otimes \mathcal{P} + \mathcal{P} \otimes \mathcal{C}, \\ [\mathcal{C}, \mathcal{P}] &= \frac{1}{2}([\mathcal{C}, \mathcal{P}] \otimes \mathbb{I} + \mathbb{I} \otimes [\mathcal{C}, \mathcal{P}]). \end{aligned} \quad (45)$$

Analogously, when one takes the time reversal operator the following relations are obtained:

$$\begin{aligned} \{\mathcal{C}, \mathcal{T}\} &= \mathcal{C} \otimes \mathcal{T} + \mathcal{T} \otimes \mathcal{C}, \\ [\mathcal{C}, \mathcal{T}] &= \frac{1}{2}([\mathcal{C}, \mathcal{T}] \otimes \mathbb{I} + \mathbb{I} \otimes [\mathcal{C}, \mathcal{T}]). \end{aligned} \quad (46)$$

Besides, as the Dirac spinor field is not an eigenspinor of the charge conjugation operator let us analyze for instance in what aspect the DKP field generated by two Dirac spinor fields is different of a DKP field constructed by the tensor product between Dirac and Elko spinor fields. For instance, the following expressions shall be useful in the next Subsections in order to realize the signatures of Elko in the DKP fields:

$$(\mathcal{C} \otimes \mathcal{P})(\psi_1 \otimes \psi_2) = \mathcal{C}\psi_1 \otimes \mathcal{P}\psi_2, \quad (47)$$

$$(\mathcal{C} \otimes \mathcal{C})(\psi_1 \otimes \psi_2) = \mathcal{C}\psi_1 \otimes \mathcal{C}\psi_2. \quad (48)$$

For the alternating tensor product, according to Eqs. (40-42) the analogue of Eq.(33) is given by

$$\mathcal{C}^2 = \mathcal{P}^2 = \mathcal{J}^2 = \mathbb{I} \hat{\otimes} \mathbb{I}. \quad (49)$$

Hereon we shall show that this is not the case when at least one of the Dirac spinor fields in the tensor product, constituting a DKP field, is substituted by an Elko spinor field, as only the charge conjugate operator must be taken into account.

B. The tensor product between Elko and Dirac spinor fields

For the charge conjugation operator $\mathcal{C} \in \text{End}(\mathcal{D} \otimes \mathcal{E})$ defined by (31) it follows that

$$\begin{aligned} \mathcal{C}^2(\psi \otimes \lambda) &= \frac{1}{2}(\mathcal{C}^2\psi \otimes \lambda + 2\mathcal{C}\psi \otimes \mathcal{C}\lambda + \psi \otimes \mathcal{C}^2\lambda) \\ &= \mathcal{C}\psi \otimes \mathcal{C}\lambda + \psi \otimes \lambda \\ &= \pm \mathcal{C}\psi \otimes \lambda + \psi \otimes \lambda, \end{aligned} \quad (50)$$

since $\mathcal{C}_{\mathcal{D}}^2 = \mathbb{I}_{\mathcal{D}}$ and $\mathcal{C}_{\mathcal{E}}^2 = \mathbb{I}_{\mathcal{E}}$. The sign in the above equation regards $\mathcal{C}\lambda^S = +\lambda^S$ and $\mathcal{C}\lambda^A = -\lambda^A$.

When we take into account a DKP field $\lambda \otimes \psi$, the action of the operator $(\mathcal{C} \otimes \mathcal{P})$ is given by

$$(\mathcal{C} \otimes \mathcal{P})(\lambda^S \otimes \psi) = +\lambda^S \otimes \mathcal{P}\psi, \quad (51)$$

$$(\mathcal{C} \otimes \mathcal{P})(\lambda^A \otimes \psi) = -\lambda^A \otimes \mathcal{P}\psi. \quad (52)$$

Comparing Eq.(47) to this case, it can be seen that Eq.(52) is a signature in the DKP field inherent to the anti self-conjugate Elko, as Eq.(51) provides a signature for the DKP field intrinsic to the Majorana field.

In addition, when the action of the operator $\mathcal{C} \otimes \mathcal{C}$ on such type of DKP fields are the following

$$(\mathcal{C} \otimes \mathcal{C})(\lambda^S \otimes \psi) = +\lambda^S \otimes \mathcal{C}\psi, \quad (53)$$

$$(\mathcal{C} \otimes \mathcal{C})(\lambda^A \otimes \psi) = -\lambda^A \otimes \mathcal{C}\psi. \quad (54)$$

As the previous case, exactly the same relations provided by Eqs.(46) hold if one substitutes the parity operator for the time reversal operator. Indeed, by taking into account the alternating tensor product we have

$$\mathcal{C}^2 = \mathbb{I} \hat{\otimes} \mathbb{I}. \quad (55)$$

Elko leaves a signature on the action of discrete symmetries on DKP fields that neither Dirac spinors nor Majorana ones can carry, as we can see for instance from Eq.(54). These results evince that the DKP field constructed by a tensor product between Dirac and Elko spinor fields is not invariant under the full Lorentz group. It shows how the Elko field induces the VSR symmetry on the DKP field. A similar case is analyzed in the next Subsection.

C. The tensor product between Elko spinor fields

The DKP algebra generated by the alternate tensor product of two Elko reflects the same properties of discrete symmetries acting on Elko spinor fields. As $\mathcal{C}\lambda^S = +\lambda^S$ and $\mathcal{C}\lambda^A = -\lambda^A$ the only possibility is to act such operator on Elko, to test the signature of Elko spinor fields on the DKP field

$$\begin{aligned} (\mathcal{C} \otimes \mathcal{C})(\lambda^S \otimes \lambda^S) &= \lambda^S \otimes \lambda^S \\ (\mathcal{C} \otimes \mathcal{C})(\lambda^A \otimes \lambda^S) &= -\lambda^A \otimes \lambda^S \\ (\mathcal{C} \otimes \mathcal{C})(\lambda^S \otimes \lambda^A) &= -\lambda^S \otimes \lambda^A \\ (\mathcal{C} \otimes \mathcal{C})(\lambda^A \otimes \lambda^A) &= \lambda^A \otimes \lambda^A \end{aligned} \quad (56)$$

and therefore the self or anti-selfconjugacy of the Elko spinor fields further influence in the action of the operator $(\mathcal{C} \otimes \mathcal{C})$ on the DKP field, leading to a typical trace.

V. FINAL REMARKS: VSR STRUCTURE IN THE DKP FIELDS

Clifford algebras play a prominent role on the construction of DKP fields. Dirac spinor fields are regular under Lounesto spinor field classification, and they do not bring any novelty about the subgroups of the Lorentz group, in the construction of the DKP fields. On the another hand, Elko spinor fields are used to incorporate the VSR symmetries, departing from the Lorentz paradigm. Cohen and Glashow have shown that time dilation, the law of velocity addition, and the universal and isotropic velocity, do not demand the entire Lorentz group. Instead, those properties can be

evinced by VSR subgroups [3]. If any of the discrete symmetries of \mathcal{P} , \mathcal{T} , \mathcal{CP} , or \mathcal{CT} is violated, the symmetry group is isomorphic to VSR subgroups and the largest one is obtained by adjoining the four spacetime translation generators to the 4-parameter subgroup $\text{SIM}(2)$.

We proved that \mathcal{C} , \mathcal{P} , and \mathcal{T} symmetries are induced on DKP fields, from Dirac spinor fields, by Eq.(49), naturally. In this way the correct properties of DKP fields can be highlighted. Some similar behavior occurs for Elko spinor fields. The DKP fields in the spaces $\mathcal{D} \hat{\otimes} \mathcal{E}$ and $\mathcal{E} \hat{\otimes} \mathcal{D}$ induce the charge conjugation operator to be an involution (Eq.(55)).

As Elko spinors are ruled by VSR $\text{HOM}(2)$ and $\text{SIM}(2)$ symmetries, and as the DKP algebra is a subalgebra in $\mathbb{C}\ell_{1,3}$, the results obtained here reflect exactly that the parity symmetry is broken, and then the standard DKP algebra is not recovered. It indicates a manifestation of the Elko symmetries on the spin-0 and spin-1 bosonic sectors of DKP algebra. DKP fields can be constructed, in this structure, by the tensor product of spinor fields. When Elko spinors are taken into account in this construction, DKP fields are shown to manifest a different signature, under the action of operators constructed from the tensor product between charge conjugation operator (in every sector of the DKP resulting field) and the parity operator (concerning the Dirac part of the DKP field). Moreover, it would be interesting to investigate this connection in other dimensions, in particular, in the plane as well as in higher dimensions [49–51].

The main idea here studied was that since $\psi \otimes \lambda$ and $\lambda \otimes \lambda'$ have different discrete symmetries to $\psi \otimes \psi'$, the vector fields associated with the former will be physically distinct from the later. In a broader sense, the mathematization previously exposed is an attempt to explore the well known richness of couplings within the DKP scope [52] in the light of VSR subgroups. In this context, the appearance and formalization of Elko spinors as a building block of the DKP resulting field is particularly suggestive.

We finalize by stressing that it is also interesting to consider the tensor product of the type $\Theta\phi^* \otimes \phi$, as denoted in Section III and show that this field have a well-defined kinematics that is different from the usual Lorentz-invariant vector field, what shall be addressed in a future publication.

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